Estimating the metric in curved spacetime with quantum fields

G. J. Milburn, and T. Downes

Citation: AIP Conference Proceedings **1363**, 107 (2011); doi: 10.1063/1.3630158

View online: https://doi.org/10.1063/1.3630158

View Table of Contents: http://aip.scitation.org/toc/apc/1363/1

Published by the American Institute of Physics

Estimating the metric in curved spacetime with quantum fields.

G. J. Milburn and T. Downes

School of Mathematics and Physics, The University of Queensland, QLD 4072, Australia

Abstract. The geometry of space-time is determined by physical measurements made with clocks and rulers. In so far as these are physical systems, the ultimate accuracy achievable is determined by quantum mechanics. In this paper we use methods from quantum parameter estimation theory to obtain uncertainty principles constraining how well we can estimate the components of a metric tensor using quantum field states propagating in curved space-time, which is treated entirely classically.

Keywords: quantum, measurement, fields
PACS: 04.62.+v,03.65.Ta

INTRODUCTION

The study of quantum fields in curved space-time is well established [1]. Typically this work is concerned with thermal states generated form the vacuum by space-time curvature, e.g. Hawking radiation. Optimal parameter estimation however requires that we consider a more general class of states and also a more general class of measurements than simple quantum counting measurements.

Salecker and Wigner[2] were the first to discuss the possible limits that quantum mechanics might place on the measurement of space-time distances. Their analysis was based on reducing all such measurements to measurements of time intervals and might best be described as semiclassical: all light signals were treated classically and only matter (in particular a variety of physical clock models) are treated using non relativistic quantum mechanics. They treated a number of examples that gave a lower bound to the mass of the clock required to achieve a given inaccuracy. The problem was taken up by Ng and van Dam[3] who also considered what would happen if the clock became so massive as to significantly change the geometry of space-time in its vicinity.

Quantum parameter estimation theory, using the quantized electro-magnetic field, was used by Caves et al.[4] to derive an uncertainty principle for estimating a space-time translation in Lorentzian space-time. The results show that the estimate may be made more accurate if the uncertainty in the number operator for one or more field modes is made very large. Such states have very large energy and momentum so in an extreme limit this will mean that one can no longer neglect the back action of the field state on the the gravitational field itself.

The metric relates proper time and proper length measurements to an arbitrarily chosen coordinate system or, more generally, the invariant interval ds^2 between nearby events. In a coordinate system x^μ it is given by $ds^2 = g_{\mu\nu}dx^\mu dx^{nu}$. The coordinate sys-

Quantum Communication, Measurement and Computing (QCMC)
AIP Conf. Proc. 1363, 107-112 (2011); doi: 10.1063/1.3630158

tem is fixed and arbitrary but the metric is determined by physical measurements using clocks and rulers. In this paper we seek to understand how quantum limits to measurement lead to uncertainties in assigning a metric. As Rovelli states[5], “The individual components of the metric tensor expressed in physical coordinates are measurable". Our task in this paper is to carefully define those physical coordinates using quantum fields in curved spacetime and then determine the uncertainty in the measured values of the metric tensor that arise due to the unavoidable quantum uncertainties in the fields used.

UNRUH’S METRIC UNCERTAINTY PRINCIPLE.

In [6] Unruh gave a simple derivation of an uncertainty principle relating the spatial metric component g_{xx} and the Einstein tensor component G^{xx}. Consider a rod of length L and cross sectional area, A. We will assume that the rod is part of a local reference frame and align the x direction parallel to the rod. We regard the choice of coordinate system as arbitrary but fixed. The relation between the coordinate separation of the end points and the proper length of the rod is given by the metric as

$$L = (g_{xx}\Delta x^2)^{1/2} \tag{1}$$

measurements of L are the used to infer g_{xx}. An error in these measurements, δL leads to an error in the inferred value of g_{xx} given by

$$\delta g_{xx} = 2g_{xx}(\delta L/L) \tag{2}$$

Physical rods are not rigid but subject to internal stresses. Fluctuations in these stresses, either due to classical or quantum zero point motion, lead to fluctuations in length. A number of authors[7, 8, 9] have attempted to give a generally covariant theory of elasticity going back to Weber[10] and Synge[11]. We will follow Unruh and proceed more heuristically.

Let τ be length of time it takes to make the measurement of the rod length. Suppose that over this time interval the momentum of a mass element at one end of the rod changes by δp_x (i.e perpendicular to the cross-sectional area, A). The corresponding change in the stress on that surface is

$$\delta S_x = \delta p_x/(A\tau) \tag{3}$$

The corresponding change in the stress-energy tensor of the rod is[11],

$$\delta T_{xx} = -\delta S_{xx} = -\delta p_x/(A\tau) \tag{4}$$

If we assume that the rod is cooled down to the ground state of al its collective vibrational degrees of freedom, so that the longitudinal mode along the length of the rod is close to a minimum uncertainty state, the uncertainty in our estimation of the parameter L is constrained by momentum fluctuations of this mode[4]

$$\delta L \geq \frac{\hbar}{\sqrt{4N\langle(\Delta\hat{p})^2\rangle}} \tag{5}$$

If we prepare a state with very large momentum fluctuations, the stress energy tensor describing the elastic properties of the rod will also fluctuate. We then set $\delta p_x = \Delta\hat{p}$ and using Eq. (4) with the uncertainty principle, Eq.(5), and Eq. (2) we have that

$$\delta g_{xx}\delta T^{xx} \geq \frac{\hbar}{V^{(4)}} \tag{6}$$

where the four-volume is defined by

$$V^{(4)} = AL\tau \tag{7}$$

and we have used $\delta T^{xx} = (g_{xx})^{-1}\delta T_{xx}$. The quantity δT^{xx} is determined by the particular physical arrangement that the experimentalist choses to measure length and Eq.(6) indicates that a very good determination of the corresponding metric component requires the fluctuations in the stress-energy tensor to be very large.

Equation 6 is the primary result, but we can go further using the Einstein equations that relate the stress energy tensor to curvature. Classically, $G_{xx} = 8\pi T_{xx}$, so fluctuations in T_{xx} imply fluctuations in the Einstein tensor as $\delta G_{xx} = 8\pi\delta T_{xx}$, thus

$$\delta g_{xx}\delta G^{xx} \geq \frac{4h}{V^{(4)}} \tag{8}$$

This result seems to represent how an extremely good measurement of length would necessarily entail such large fluctuations in the stress energy tensor that the space time geometry itself would exhibit fluctuations. However as, Unruh points out, compared to the usual uncertainty principle in non relativistc quantum mechanics this relation is unusual as $G_{\mu\nu}$ in general involves second order derivatives of the metric components.

QUANTUM PARAMETER ESTIMATION VIA QUANTUM FIELDS.

The determination of temporal duration and spatial translation require us to sample a probability distribution by making measurements on an ensemble of identically prepared physical systems. This scenario leads directly to parameter based uncertainty relations of Braunstein and Caves[12]. Consider a one parameter unitary transformations

$$\rho_0 \to \rho_X = e^{-iX\hat{G}}\rho e^{iX\hat{G}}. \tag{9}$$

The objective is to find a lower bound to the mean square deviation, ΔX^2, of the estimate optimised over all generalised measurements, Braunstein and Caves showed that[12]

$$\overline{\Delta X^2} \geq \frac{1}{F(\xi)} \geq \frac{1}{4\langle(\Delta G)^2\rangle_0} \tag{10}$$

where $F(\xi)$ is the Fisher information of the distribution of measurement results.

Quantum parameter estimation in flat space time using quantum fields were considered in [4]. This can be done to arbitrary precision given sufficient energy. This implies that the ultimate limits necessarily requires us to consider curved space time. As many

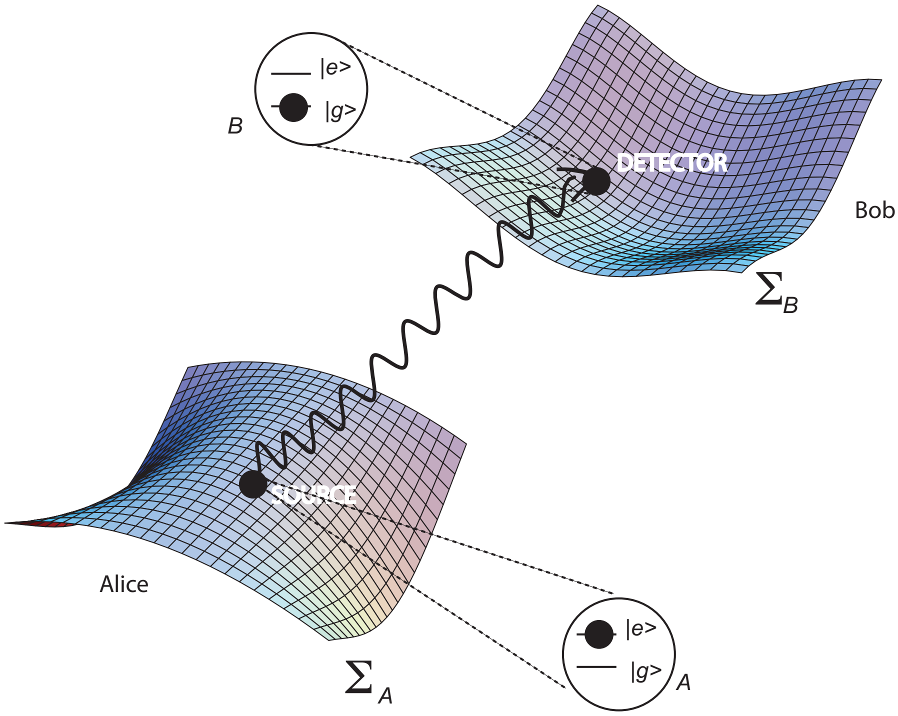

FIGURE 1. A scenario using quantum fields to estimate the curvature of spacetime.

authors have pointed out, if we push to very high energy to improve accuracy we will change the geometry of spacetime, culminating in the extreme case of producing a black hole.

The case of curved spacetime presents some difficulties. First of all general covariance means the coordinates (t,x,y,z) are not physical. Thus our estimation protocol must refer to physical facts such as the preparation of sources and the response of detectors at distinct spacetime points. Second, quantum fields in curved spacetime have no preferred vacuum state and thus there is no natural notion of particles. As Wald[13] says, *"the notion of particle plays no fundamental role either in the formulation or the interpretation of the theory"*. This problem can also be avoided to some extent by explicitly formulating the protocol in terms of physical sources and detectors. We thus need to consider a scenario something like what is depicted in Fig.1. The objective is to measure local metric components by making measurements on the field, optimised over the source and detector. This is a general relativistic version of the relativistic quantum channel model considered by Cliche and Kempf[14] . The nature of the metric is encoded in the propagator between source and detector and it is this that determines the probability for the detector to become excited.

Assume massless scalar field which obeys the *classical wave equation* in curved spacetime

$$(-g)^{1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\phi]+\frac{1}{6}R(x)\phi=0 \tag{11}$$

where R Ricci scalar curvature of metric $g_{\mu\nu}$, g is the determinant of the metric. We can find a complete set of solutions, $u_n^{(A)}(x)$, that satisfy the wave equation, but these are *not* unique positive frequency modes.

Given a complete set of modes, we expand the quantum field

$$\hat{\phi}(x) = \sum_n a_n u_n^{(A)}(x) + a_n^\dagger u_n^{(A)*}(x) \tag{12}$$

with $[a_n, a_m^\dagger] = \delta_{nm}$ and define a vacuum state with respect to the A modes $a_n|0\rangle_A = 0$. We can equally well expand in terms of another set of modes,

$$\hat{\phi}(x) = \sum_n b_n u_n^{(B)}(x) + b_n^\dagger u_n^{(B)*}(x) \tag{13}$$

with $[b_n, b_m^\dagger] = \delta_{nm}$ and define a vacuum state with respect to the B-modes $b_n|0\rangle_B = 0$. The two sets of mode operators are related by a Bogoluibov transformation

$$\begin{aligned} a_n &= \sum_m \alpha_{nm} b_m + \beta_{nm}^* b_m^\dagger \\ b_n &= \sum_m \alpha_{nm}^* a_m - \beta_{nm} a_m^\dagger \end{aligned}$$

Then,

$$_B\langle 0|a_n^\dagger a_n|0\rangle_B = \sum_n ||\beta_{nm}||^2 \neq 0 \tag{14}$$

Here is the problem: the vacuum states for one mode expansion is not the vacuum state for another mode expansion.

In figure 1 we use two-level systems with energy eigenstates $|g\rangle, |e\rangle$. The interaction Hamiltonian between the source and detector is is,

$$H_I(t) = \varepsilon(t) \int d\vec{x} f(\vec{x}) \hat{\phi}(\vec{x}, t) \sigma_x$$

where $\varepsilon(t)$ is smoothly turned on and off at finite times. We assume that the source at A is excited, source at B is not. The objective is to compute the probability to make the transition $|e\rangle_A, |g\rangle_B \rightarrow |g\rangle_A |e\rangle_B$. This will depend on the metric through the field propagator. In this context the problem of no unique vacuum state means there is some ambiguity as to how the problem is stated. Detectors and sources always couple locally to the field however the local field modes at the source are not necessarily the same as those at the detector. If we assumed that the detector coupled to the vacuum defined by the modes at the source the detector would fire regardless of weather or not the source was present. To avoid that possibility, we need to use Alice's vacuum state $|0\rangle_A$ defined by the mode functions relevant for the source construction, but expand the field in the modes defined by how the detector is constructed. This means that the fundamental probability is not simply determined by the Feynman propagator for the field as it wold be in flat spacetime. This leads to some technical difficulties that might be addressed using techniques from algebraic quantum field theory.

REFERENCES

1. N. Birrell and P. C. W. Davies, *Quantum fields in curved spacetime*, (Cambridge University press, 1982).
2. H. Salecker and E. P. Wigner, *Quantum limitations of the measurement if space-time distances*, Phys. Rev. **109**, 571 (1958).
3. Y. J. Ng and H. van Dam, Annals N. Y. Acad. Sci. **755**, 579 (1995).
4. 83. S.L.Braunstein , C.M. Caves and G.J.Milburn, *Generalized uncertainty relations: Theory, examples and Lorentz Invariance*, Annals of Physics, 247, 135-173, number 1, (1996).
5. C. Rovelli, Quantum Gravity, Cambridge University Press, ().
6. W. G. Unruh, Why study quantum theory?, Canadian J. Phys. **64**, 128 (1986).
7. G. A. Maugin, *Harmonic oscillations of elastic contiua and detection of gravitational waves*, Gen. Rel. Grav. **4**, 241 (1973).
8. G. A. Maugin, *Infinitesimal discontinuities in initially stressed relativistic solids*, Comm. Math. Phys. **53**, 233 (1977).
9. J. Kijowski and G. Magli, *Relativistic elastomechanics as a Lagrangian field theory*, J. geometry and Physics, **9**, 207 (1992).
10. J. Weber, *Detection and generation of gravitational waves*, Physical Review, **117**, 306 (1960).
11. J. L. Synge, *A theory of elasticity in general relativity*, Math. Zeitschr, **72**, 82 (1959).
12. S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439, (1994).
13. R. M. Wald, *Quantum Field theory in curved spacetime and black hole thermodynamics*, (1994).
14. Cliche, M.; Kempf, A., Phys. Rev. A, **81**, 012330 (2010)